

# On the Complex Symmetric and Skew-Symmetric Operators with a Simple Spectrum

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**Abstract.** In this paper we obtain necessary and sufficient conditions for a linear bounded operator in a Hilbert space  $H$  to have a three-diagonal complex symmetric matrix with non-zero elements on the first sub-diagonal in an orthonormal basis in  $H$ . It is shown that a set of all such operators is a proper subset of a set of all complex symmetric operators with a simple spectrum. Similar necessary and sufficient conditions are obtained for a linear bounded operator in  $H$  to have a three-diagonal complex skew-symmetric matrix with non-zero elements on the first sub-diagonal in an orthonormal basis in  $H$ .

*Key words:* complex symmetric operator; complex skew-symmetric operator; cyclic operator; simple spectrum

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## 1 Introduction

In last years an increasing interest is devoted to the subject of operators related to bilinear forms in a Hilbert space (see [1, 2, 3] and references therein), i.e. to the following forms:

$$[x, y]_J := (x, Jy)_H, \quad x, y \in H,$$

where  $J$  is a conjugation and  $(\cdot, \cdot)_H$  is the inner product in a Hilbert space  $H$ . The conjugation  $J$  is an *antilinear* operator in  $H$  such that  $J^2x = x$ ,  $x \in H$ , and

$$(Jx, Jy)_H = (y, x)_H, \quad x, y \in H.$$

Recall that a linear operator  $A$  in  $H$  is said to be  $J$ -symmetric ( $J$ -skew-symmetric) if

$$[Ax, y]_J = [x, Ay]_J, \quad x, y \in D(A), \tag{1}$$

or, respectively,

$$[Ax, y]_J = -[x, Ay]_J, \quad x, y \in D(A). \tag{2}$$

If a linear bounded operator  $A$  in a Hilbert space  $H$  is  $J$ -symmetric ( $J$ -skew-symmetric) for a conjugation  $J$  in  $H$ , then  $A$  is said to be complex symmetric (respectively complex skew-symmetric). The matrices of complex symmetric (skew-symmetric) operators in certain bases of  $H$  are complex symmetric (respectively skew-symmetric) semi-infinite matrices. Observe that for a bounded linear operator  $A$  conditions (1) and (2) are equivalent to conditions

$$JAJ = A^*, \tag{3}$$

and

$$JAJ = -A^*,$$

respectively.

Recall that a bounded linear operator  $A$  in a Hilbert space  $H$  is said to have a simple spectrum if there exists a vector  $x_0 \in H$  (cyclic vector) such that

$$\overline{\text{Lin}\{A^k x_0, k \in \mathbb{Z}_+\}} = H.$$

Observe that these operators are also called *cyclic operators*.

It is well known that a bounded self-adjoint operator with a simple spectrum has a bounded semi-infinite real symmetric three-diagonal (Jacobi) matrix in a certain orthonormal basis (e.g. [4, Theorem 4.2.3]).

The aim of our present investigation is to describe a class  $C_+ = C_+(H)$  ( $C_- = C_-(H)$ ) of linear bounded operators in a Hilbert space  $H$ , which have three-diagonal complex symmetric (respectively skew-symmetric) matrices with non-zero elements on the first sub-diagonal in some orthonormal bases of  $H$ . We obtain necessary and sufficient conditions for a linear bounded operator in a Hilbert space  $H$  to belong to the class  $C_+$  ( $C_-$ ). The class  $C_+$  ( $C_-$ ) is a subset of the class of all complex symmetric (respectively skew-symmetric) operators in  $H$  with a simple spectrum. Moreover, it is shown that  $C_+(H)$  is a proper subset.

**Notations.** As usual, we denote by  $\mathbb{R}$ ,  $\mathbb{C}$ ,  $\mathbb{N}$ ,  $\mathbb{Z}$ ,  $\mathbb{Z}_+$  the sets of real numbers, complex numbers, positive integers, integers and non-negative integers, respectively;  $\text{Im } z = \frac{1}{2i}(z - \bar{z})$ ,  $z \in \mathbb{C}$ . Everywhere in this paper, all Hilbert spaces are assumed to be separable. By  $(\cdot, \cdot)_H$  and  $\|\cdot\|_H$  we denote the scalar product and the norm in a Hilbert space  $H$ , respectively. The indices may be omitted in obvious cases. For a set  $M$  in  $H$ , by  $\overline{M}$  we mean the closure of  $M$  in the norm  $\|\cdot\|_H$ . For  $\{x_n\}_{n \in \mathbb{Z}_+}$ ,  $x_n \in H$ , we write  $\text{Lin}\{x_n\}_{n \in \mathbb{Z}_+}$  for the set of linear combinations of elements  $\{x_n\}_{n \in \mathbb{Z}_+}$ . The identity operator in  $H$  is denoted by  $E_H$ . For an arbitrary linear operator  $A$  in  $H$ , the operators  $A^*$ ,  $\overline{A}$ ,  $A^{-1}$  mean its adjoint operator, its closure and its inverse (if they exist). By  $D(A)$  and  $R(A)$  we mean the domain and the range of the operator  $A$ . The norm of a bounded operator  $A$  is denoted by  $\|A\|$ . By  $P_{H_1}^H = P_{H_1}$  we mean the operator of orthogonal projection in  $H$  on a subspace  $H_1$  in  $H$ .

## 2 The classes $C_{\pm}(H)$

Let  $\mathcal{M} = (m_{k,l})_{k,l=0}^{\infty}$ ,  $m_{k,l} \in \mathbb{C}$ , be a semi-infinite complex matrix. We shall say that  $\mathcal{M}$  belongs to the class  $\mathfrak{M}_3^+$ , if and only if the following conditions hold:

$$m_{k,l} = 0, \quad k, l \in \mathbb{Z}_+, \quad |k - l| > 1, \quad (4)$$

$$m_{k,l} = m_{l,k}, \quad k, l \in \mathbb{Z}_+, \quad (5)$$

$$m_{k,k+1} \neq 0, \quad k \in \mathbb{Z}_+. \quad (6)$$

We shall say that  $\mathcal{M}$  belongs to the class  $\mathfrak{M}_3^-$ , if and only if the conditions (4), (6) hold and

$$m_{k,l} = -m_{l,k}, \quad k, l \in \mathbb{Z}_+.$$

Let  $A$  be a linear bounded operator in an infinite-dimensional Hilbert space  $H$ . We say that  $A$  belongs to the class  $C_+ = C_+(H)$  ( $C_- = C_-(H)$ ) if and only if there exists an orthonormal basis  $\{e_k\}_{k=0}^{\infty}$  in  $H$  such that the matrix

$$\mathcal{M} = ((Ae_l, e_k))_{k,l=0}^{\infty}, \quad (7)$$

belongs to  $\mathfrak{M}_3^+$  (respectively to  $\mathfrak{M}_3^-$ ).

Let  $y_0, y_1, \dots, y_n$  be arbitrary vectors in  $H$ ,  $n \in \mathbb{Z}_+$ . Set

$$\Gamma(y_0, y_1, \dots, y_n) := \det((y_k, y_l)_H)_{k,l=0}^n.$$

Thus,  $\Gamma(y_0, y_1, \dots, y_n)$  is the Gram determinant of vectors  $y_0, y_1, \dots, y_n$ .

The following theorem provides a description of the class  $C_+(H)$ .

**Theorem 1.** *Let  $A$  be a linear bounded operator in an infinite-dimensional Hilbert space  $H$ . The operator  $A$  belongs to the class  $C_+(H)$  if and only if the following conditions hold:*

- (i)  *$A$  is a complex symmetric operator with a simple spectrum;*
- (ii) *there exists a cyclic vector  $x_0$  of  $A$  such that the following relations hold:*

$$\Gamma(x_0, x_1, \dots, x_n, x_n^*) = 0, \quad \forall n \in \mathbb{N}, \quad (8)$$

where

$$x_k = A^k x_0, \quad x_k^* = (A^*)^k x_0, \quad k \in \mathbb{N};$$

and  $Jx_0 = x_0$ , for a conjugation  $J$  in  $H$  such that  $JAJ = A^*$ .

**Proof.** *Necessity.* Let  $H$  be an infinite-dimensional Hilbert space and  $A \in C_+(H)$ . Let  $\{e_k\}_{k=0}^\infty$  be an orthonormal basis in  $H$  such that the matrix  $\mathcal{M} = (m_{k,l})_{k,l=0}^\infty$  belongs to  $\mathfrak{M}_3^+$ , where  $m_{k,l} = ((Ae_l, e_k))_{k,l=0}^\infty$ . Observe that

$$\begin{aligned} Ae_0 &= m_{0,0}e_0 + m_{1,0}e_1, \\ Ae_k &= m_{k-1,k}e_{k-1} + m_{k,k}e_k + m_{k+1,k}e_{k+1}, \quad k \in \mathbb{N}. \end{aligned} \quad (9)$$

Suppose that

$$e_r \in \text{Lin}\{A^j e_0, 0 \leq j \leq r\}, \quad 0 \leq r \leq n,$$

for some  $n \in \mathbb{N}$  (for  $n = 0$  it is trivial). By (9) we may write

$$e_{n+1} = \frac{1}{m_{n+1,n}} (Ae_n - m_{n-1,n}e_{n-1} - m_{n,n}e_n) \in \text{Lin}\{A^j e_0, 0 \leq j \leq n+1\}.$$

Here  $m_{-1,0} := 0$  and  $e_{-1} := 0$ . By induction we conclude that

$$e_r \in \text{Lin}\{A^j e_0, 0 \leq j \leq r\}, \quad r \in \mathbb{Z}_+. \quad (10)$$

Therefore  $\overline{\text{Lin}\{A^j e_0, j \in \mathbb{Z}_+\}} = H$ , i.e. the operator  $A$  has a simple spectrum and  $e_0$  is a cyclic vector of  $A$ .

Consider the following conjugation:

$$J \sum_{k=0}^\infty x_k e_k = \sum_{k=0}^\infty \overline{x_k} e_k, \quad x = \sum_{k=0}^\infty x_k e_k \in H.$$

Observe that

$$[Ae_k, e_l]_J = (Ae_k, e_l) = m_{l,k} = m_{k,l} = (Ae_l, e_k) = [Ae_l, e_k]_J, \quad k, l \in \mathbb{Z}_+.$$

By linearity of the  $J$ -form  $[\cdot, \cdot]_J$  in the both arguments we get

$$[Ax, y]_J = [Ay, x]_J, \quad x, y \in H.$$

Thus, the operator  $A$  is  $J$ -symmetric and relation (3) holds. Notice that  $Je_0 = e_0$ . It remains to check if relation (8) holds. Set

$$H_r := \text{Lin}\{A^j e_0, 0 \leq j \leq r\}, \quad r \in \mathbb{Z}_+.$$

By (10) we see that  $e_0, e_1, \dots, e_r \in H_r$ , and therefore  $\{e_j\}_{j=0}^r$  is an orthonormal basis in  $H_r$  ( $r \in \mathbb{Z}_+$ ). Since  $Je_j = e_j$ ,  $j \in \mathbb{Z}_+$ , we have

$$JH_r \subseteq H_r, \quad r \in \mathbb{Z}_+.$$

Then

$$(A^*)^r e_0 = (JAJ)^r e_0 = JA^r Je_0 = JA^r e_0 \in H_r, \quad r \in \mathbb{Z}_+.$$

Therefore vectors  $e_0, Ae_0, \dots, A^r e_0, (A^*)^r e_0$ , are linearly dependent and their Gram determinant is equal to zero. Thus, relation (8) holds with  $x_0 = e_0$ .

*Sufficiency.* Let  $A$  be a bounded operator in a Hilbert space  $H$  satisfying conditions (i), (ii) in the statement of the theorem. For the cyclic vector  $x_0$  we set

$$H_r := \text{Lin}\{A^j x_0, 0 \leq j \leq r\}, \quad r \in \mathbb{Z}_+.$$

Observe that

$$A^{r+1}x_0 \notin H_r, \quad r \in \mathbb{Z}_+. \quad (11)$$

In fact, suppose that for some  $k \in \mathbb{N}$ , we have

$$A^{r+j}x_0 \in H_r, \quad 1 \leq j \leq k.$$

Then

$$A^{r+k+1}x_0 = AA^{r+k}x_0 = A \sum_{t=0}^r \alpha_{r,k;t} A^t x_0 = \sum_{t=0}^r \alpha_{r,k;t} A^{t+1} x_0 \in H_r, \quad \alpha_{r,k;t} \in \mathbb{C}.$$

By induction we obtain

$$A^{r+j}x_0 \in H_r, \quad j \in \mathbb{Z}_+.$$

Therefore  $H = H_r$ . We obtain a contradiction since  $H$  is infinite-dimensional.

Let us apply the Gram–Schmidt orthogonalization method to the sequence  $x_0, Ax_0, A^2x_0, \dots$ . Namely, we set

$$g_0 = \frac{x_0}{\|x_0\|_H}, \quad g_{r+1} = \frac{A^{r+1}x_0 - \sum_{j=0}^r (A^{r+1}x_0, g_j)_H g_j}{\left\| A^{r+1}x_0 - \sum_{j=0}^r (A^{r+1}x_0, g_j)_H g_j \right\|_H}, \quad r \in \mathbb{Z}_+.$$

By construction we have

$$H_r = \text{Lin}\{g_j, 0 \leq j \leq r\}, \quad r \in \mathbb{Z}_+.$$

Therefore  $\{g_j\}_{j=0}^r$  is an orthonormal basis in  $H_r$  ( $r \in \mathbb{Z}_+$ ) and  $\{g_j\}_{j \in \mathbb{Z}_+}$  is an orthonormal basis in  $H$ .

From (8) and (11) we conclude that

$$JA^n x_0 = JA^n Jx_0 = (A^*)^n x_0 \in H_n, \quad n \in \mathbb{Z}_+.$$

Therefore

$$JH_r \subseteq H_r, \quad r \in \mathbb{Z}_+. \quad (12)$$

Let

$$Jg_r = \sum_{j=0}^r \beta_{r,j} g_j, \quad \beta_{r,j} \in \mathbb{C}, \quad r \in \mathbb{Z}_+.$$

Using properties of the conjugation and relation (12) we get

$$\beta_{r,j} = (Jg_r, g_j)_H = (Jg_r, JJg_j)_H = \overline{(g_r, Jg_j)_H} = 0,$$

for  $0 \leq j \leq r-1$ . Therefore

$$Jg_r = \beta_{r,r} g_r, \quad \beta_{r,r} \in \mathbb{C}, \quad r \in \mathbb{Z}_+.$$

Since  $\|g_r\|^2 = \|Jg_r\|^2 = |\beta_{r,r}|^2 \|g_r\|^2$ , we have

$$\beta_{r,r} = e^{i\varphi_r}, \quad \varphi_r \in [0, 2\pi), \quad r \in \mathbb{Z}_+.$$

Set

$$e_r := e^{i\frac{\varphi_r}{2}} g_r, \quad r \in \mathbb{Z}_+.$$

Then  $\{e_j\}_{j=0}^r$  is an orthonormal basis in  $H_r$  ( $r \in \mathbb{Z}_+$ ) and  $\{e_j\}_{j \in \mathbb{Z}_+}$  is an orthonormal basis in  $H$ . Observe that

$$Je_r = Je^{i\frac{\varphi_r}{2}} g_r = e^{-i\frac{\varphi_r}{2}} Jg_r = e^{i\frac{\varphi_r}{2}} g_r = e_r, \quad r \in \mathbb{Z}_+.$$

Define the matrix  $\mathcal{M} = (m_{k,l})_{k,l=0}^\infty$  by (7). Notice that

$$m_{k,l} = (Ae_l, e_k)_H = [Ae_l, e_k]_J = [e_l, Ae_k]_J = [Ae_k, e_l]_J = (Ae_k, e_l)_H = m_{l,k},$$

where  $k, l \in \mathbb{Z}_+$ , and therefore  $\mathcal{M}$  is complex symmetric.

If  $l \geq k+2$  ( $k, l \in \mathbb{Z}_+$ ), then

$$m_{k,l} = (Ae_l, e_k)_H = [Ae_l, e_k]_J = [e_l, Ae_k]_J = (e_l, JAe_k)_H = 0,$$

since  $JAe_k \in H_{k+1} \subseteq H_{l-1}$ , and  $e_l \in H_l \ominus H_{l-1}$ . Therefore  $\mathcal{M}$  is three-diagonal.

Since  $e_r \in H_r$ , using the definition of  $H_r$  we get

$$Ae_r \subseteq H_{r+1}, \quad r \in \mathbb{Z}_+.$$

Observe that

$$Ae_r \notin H_r, \quad r \in \mathbb{Z}_+.$$

In fact, in the opposite case we get

$$Ae_j \in H_r, \quad 0 \leq j \leq r,$$

and  $AH_r \subseteq H_r$ . Then  $A^k x_0 \in H_r$ ,  $k \in \mathbb{Z}_+$ , and  $H = H_r$ . This is a contradiction since  $H$  is an infinite-dimensional space.

Hence, we may write

$$Ae_r = \sum_{j=0}^{r+1} \gamma_{r,j} e_j, \quad \gamma_{r,j} \in \mathbb{C}, \quad \gamma_{r,r+1} \neq 0.$$

Observe that

$$m_{r+1,r} = (Ae_r, e_{r+1})_H = \gamma_{r,r+1} \neq 0, \quad r \in \mathbb{Z}_+.$$

Thus,  $\mathcal{M} \in \mathfrak{M}_3^+$  and  $A \in C_+(H)$ . ■

**Remark 1.** Condition (ii) of the last theorem may be replaced by the following condition which does not use a conjugation  $J$ :

(ii)\* there exists a cyclic vector  $x_0$  of  $A$  such that the following relations hold:

$$\Gamma(x_0, x_1, \dots, x_n, x_n^*) = 0, \quad \forall n \in \mathbb{N}, \quad (13)$$

where

$$x_k = A^k x_0, \quad x_k^* = (A^*)^k x_0, \quad k \in \mathbb{N},$$

and the following operator:

$$L \sum_{k=0}^{\infty} \alpha_k A^k x_0 := \sum_{k=0}^{\infty} \overline{\alpha_k} (A^*)^k x_0, \quad \alpha_k \in \mathbb{C}, \quad (14)$$

where all but finite number of coefficients  $\alpha_k$  are zeros, is a bounded operator in  $H$  which extends by continuity to a conjugation in  $H$ .

Let us show that conditions (i), (ii)  $\Leftrightarrow$  conditions (i), (ii)\*.

The necessity is obvious since the conjugation  $J$  satisfies relation (14) (with  $J$  instead of  $L$ ).

*Sufficiency.* Let conditions (i), (ii)\* be satisfied. Notice that

$$\begin{aligned} L A A^k x_0 &= L A^{k+1} x_0 = (A^*)^{k+1} x_0, \\ A^* L A^k x_0 &= A^* (A^*)^k x_0 = (A^*)^{k+1} x_0, \quad k \in \mathbb{Z}_+. \end{aligned}$$

By continuity we get  $LA = A^*L$ . Then condition (ii) holds with the conjugation  $L$ .

**Remark 2.** Notice that conditions (13) may be written in terms of the coordinates of  $x_0$  in an arbitrary orthonormal basis  $\{u_n\}_{n=0}^{\infty}$  in  $H$ :

$$x_0 = \sum_{n=0}^{\infty} x_{0,n} u_n, \quad A^k x_0 = \sum_{n=0}^{\infty} x_{0,n} A^k u_n, \quad (A^*)^k x_0 = \sum_{n=0}^{\infty} x_{0,n} (A^*)^k u_n.$$

By substitution these equalities in relation (8) we get some algebraic equations with respect to the coordinates  $x_{0,n}$ . If cyclic vectors of  $A$  are unknown, one can use numerical methods to find approximate solutions of these equations. Then there should be cyclic vectors of  $A$  among these solutions.

The following theorem gives an analogous description for the class  $C_-(H)$ .

**Theorem 2.** Let  $A$  be a linear bounded operator in an infinite-dimensional Hilbert space  $H$ . The operator  $A$  belongs to the class  $C_-(H)$  if and only if the following conditions hold:

- (i)  $A$  is a complex skew-symmetric operator with a simple spectrum;
- (ii) there exists a cyclic vector  $x_0$  of  $A$  such that the following relations hold:

$$\Gamma(x_0, x_1, \dots, x_n, x_n^*) = 0, \quad \forall n \in \mathbb{N}, \quad (15)$$

where

$$x_k = A^k x_0, \quad x_k^* = (A^*)^k x_0, \quad k \in \mathbb{N}; \quad (16)$$

and  $Jx_0 = x_0$ , for a conjugation  $J$  in  $H$  such that  $JAJ = -A^*$ .

Condition (ii) of this theorem may be replaced by the following condition:

(ii)\* there exists a cyclic vector  $x_0$  of  $A$  such that relations (15), (16) hold and the following operator:

$$L \sum_{k=0}^{\infty} \alpha_k A^k x_0 := \sum_{k=0}^{\infty} (-1)^k \overline{\alpha_k} (A^*)^k x_0, \quad \alpha_k \in \mathbb{C},$$

where all but finite number of coefficients  $\alpha_k$  are zeros, is a bounded operator in  $H$  which extends by continuity to a conjugation in  $H$ .

The proof of the latter facts is similar and essentially the same as for the case of  $C_+(H)$ .

The following example shows that condition (ii) (or (ii)\*) can not be removed from Theorem 1.

**Example 1.** Let  $\sigma(\theta)$  be a non-decreasing left-continuous bounded function on  $[0, 2\pi]$  with an infinite number of points of increase and such that

$$\int_0^{2\pi} \ln \sigma'(\theta) d\theta = -\infty. \quad (17)$$

Consider the Hilbert space  $L^2([0, 2\pi], d\sigma)$  of (classes of equivalence of) complex-valued functions  $f(\theta)$  on  $[0, 2\pi]$  such that

$$\|f\|_{L^2([0, 2\pi], d\sigma)}^2 := \left( \int_0^{2\pi} |f(\theta)|^2 d\sigma(\theta) \right)^{\frac{1}{2}} < \infty.$$

The condition (17) provides that algebraic polynomials of  $e^{i\theta}$  are dense in  $L^2([0, 2\pi], d\sigma)$  [5, p. 19]. Therefore the operator

$$Uf(\theta) = e^{i\theta} f(\theta), \quad f \in L^2([0, 2\pi], d\sigma),$$

is a cyclic unitary operator in  $H$ , with a cyclic vector  $f_0(\theta) = 1$ . Set

$$Jf(\theta) = \overline{f(\theta)}, \quad f \in L^2([0, 2\pi], d\sigma).$$

Then

$$JUJf(\theta) = J e^{i\theta} \overline{f(\theta)} = e^{-i\theta} f(\theta) = U^{-1}f(\theta) = U^*f(\theta).$$

Thus,  $U$  is a complex symmetric operator with a simple spectrum and condition (i) of Theorem 1 is satisfied.

However,  $U \notin C_+(H)$ . In fact, suppose to the contrary that there exists an orthonormal basis  $\{e_j\}_{j \in \mathbb{Z}_+}$  such that the corresponding matrix  $\mathcal{M} = (m_{k,l})_{k,l=0}^{\infty}$  from (7) belongs to the class  $\mathfrak{M}_3^+$ . Since  $U$  is unitary, we have

$$\mathcal{E} = \mathcal{M}\mathcal{M}^*,$$

with the usual rules of matrix operations,  $\mathcal{E} = (\delta_{k,l})_{k,l=0}^{\infty}$ . However, the direct calculation shows that the element of the matrix  $\mathcal{M}\mathcal{M}^*$  in row 0, column 2 is equal to  $m_{0,1}\overline{m_{2,1}} \neq 0$ . We obtained a contradiction. Thus,  $U \notin C_+(H)$ . Consequently, *condition (ii) in Theorem 1 is essential and can not be removed.*

**Proposition 1.** *Let  $H$  be an arbitrary infinite-dimensional Hilbert space. The class  $C_+(H)$  is a proper subset of the set of all complex symmetric operators with a simple spectrum in  $H$ .*

**Proof.** Consider an arbitrary infinite-dimensional Hilbert space  $H$ . Let  $V$  be an arbitrary unitary operator which maps  $L^2([0, 2\pi], d\sigma)$  (see Example 1) onto  $H$ . Then  $\widehat{U} := VUV^{-1}$  is a unitary operator in  $H$  with a simple spectrum and it has a cyclic vector  $\widehat{x}_0 := V1$ . Since  $JUJ = U^*$ , we get

$$JV^{-1}\widehat{U}VJ = V^{-1}\widehat{U}^*V, \quad VJV^{-1}\widehat{U}VJV^{-1} = \widehat{U}^*.$$

Observe that  $\widehat{J} := VJV^{-1}$  is a conjugation in  $H$ . Therefore  $\widehat{U}$  is a complex symmetric operator in  $H$ . Suppose that  $\widehat{U} \in C_+(H)$ . Let  $\mathcal{F} = \{f_k\}_{k=0}^\infty$  be an orthonormal basis in  $H$  such that the matrix  $M = (m_{k,l})_{k,l=0}^\infty$ ,  $m_{k,l} = (\widehat{U}f_l, f_k)_H$ , belongs to  $\mathfrak{M}_3^+$ . Observe that  $\mathcal{G} = \{g_k\}_{k=0}^\infty$ ,  $g_k := V^{-1}f_k$ , is an orthonormal basis in  $L^2([0, 2\pi], d\sigma)$  and

$$(Ug_l, g_k)_{L^2([0, 2\pi], d\sigma)} = (V^{-1}\widehat{U}Vg_l, g_k)_{L^2([0, 2\pi], d\sigma)} = (\widehat{U}f_l, f_k)_H = m_{k,l}, \quad k, l \in \mathbb{Z}_+.$$

Therefore  $U \in C_+(L^2([0, 2\pi], d\sigma))$ . This is a contradiction with Example 1. Consequently, we have  $\widehat{U} \notin C_+(H)$ .

On the other hand, the class  $C_+(H)$  is non-empty, since an arbitrary matrix from  $\mathcal{M}_3^+$  with bounded elements define an operator  $B$  in  $H$  which have this matrix in an arbitrary fixed orthonormal basis in  $H$ . ■

**Remark 3.** The classical Jacobi matrices are closely related to orthogonal polynomials [4]. Let us indicate some similar relations for the class  $\mathfrak{M}_3^+$ . Choose an arbitrary  $\mathcal{M} = (m_{k,l})_{k,l=0}^\infty \in \mathfrak{M}_3^+$ , where  $m_{k,l} \in \mathbb{C}$ . Let  $\{p_n(\lambda)\}_{n=0}^\infty$ ,  $\deg p_n = n$ ,  $p_0(\lambda) = 1$ , be a sequence of polynomials defined recursively by the following relation:

$$m_{n,n-1}p_{n-1}(\lambda) + m_{n,n}p_n(\lambda) + m_{n,n+1}p_{n+1}(\lambda) = \lambda p_n(\lambda), \quad n = 0, 1, 2, \dots, \quad (18)$$

where  $m_{0,-1} := 1$ ,  $p_{-1} := 0$ . Set  $c_n = m_{n,n+1}$ ,  $b_n = m_{n,n}$ ,  $n \in \mathbb{Z}_+$ ; and  $c_{-1} := 1$ . By (5), (18) we get

$$c_{n-1}p_{n-1}(\lambda) + b_n p_n(\lambda) + c_n p_{n+1}(\lambda) = \lambda p_n(\lambda), \quad n = 0, 1, 2, \dots \quad (19)$$

Let  $p_n(\lambda) = \mu_n \lambda^n + \dots$ ,  $\mu_n \in \mathbb{C}$ ,  $n \in \mathbb{Z}_+$ . Comparing coefficients by  $\lambda^{n+1}$  in (19) we get

$$\mu_{n+1} = \frac{1}{c_n} \mu_n, \quad n \in \mathbb{Z}_+.$$

By induction we see that

$$\mu_n = \left( \prod_{j=0}^{n-1} c_j \right)^{-1}, \quad n \in \mathbb{N}, \quad \mu_0 = 1.$$

Set

$$P_n(\lambda) = \prod_{j=0}^{n-1} c_j p_n(\lambda), \quad n \in \mathbb{N}, \quad P_0(\lambda) = 1, \quad P_{-1}(\lambda) = 0.$$

Multiplying the both sides of (19) by  $\prod_{j=0}^{n-1} c_j$ ,  $n \geq 1$ , we obtain:

$$c_{n-1}^2 P_{n-1}(\lambda) + b_n P_n(\lambda) + P_{n+1}(\lambda) = \lambda P_n(\lambda), \quad n = 0, 1, 2, \dots$$



By Theorem 6.4 in [6] there exists a complex-valued function  $\phi$  of bounded variation on  $\mathbb{R}$  such that

$$\int_{\mathbb{R}} P_m(\lambda) P_n(\lambda) d\phi(\lambda) = \left( \prod_{j=0}^{n-1} c_j \right)^2 \delta_{m,n}, \quad m, n \in \mathbb{Z}_+.$$

Therefore we get

$$\int_{\mathbb{R}} p_m(\lambda) p_n(\lambda) d\phi(\lambda) = \delta_{m,n}, \quad m, n \in \mathbb{Z}_+.$$

Polynomials  $\{p_n(\lambda)\}_{n=0}^{\infty}$  were used in [7, 8] to state and solve the direct and inverse spectral problems for matrices from  $\mathfrak{M}_3^+$ . Analogs of some facts of the Weyl discs theory were obtained for the case of matrices from  $\mathfrak{M}_3^+$  with additional assumptions [9]:  $m_{n,n+1} > 0$ ,  $n \in \mathbb{Z}_+$ , and

$$m_{n,n} \in \mathbb{C}: \quad r_0 \leq \operatorname{Im} m_{n,n} \leq r_1,$$

for some  $r_0, r_1 \in \mathbb{R}$ ,  $n \in \mathbb{Z}_+$ .

On the other hand, the direct and inverse spectral problems for matrices from  $\mathfrak{M}_3^-$  were investigated in [10].

Probably, some progress in the spectral theory of complex symmetric and skew-symmetric operators would provide some additional information about corresponding polynomials and vice versa.

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